Math Interlude! The Rate of Neutrinoless Double Beta Decay

Ben Jones, National Nuclear Physics Summer School Lecture 3

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1 Outline of decay rate calculation

Calculating the rate of either neutrinoless or two-neutrino double beta decay is a tricky business, because it involves understanding the properties of the nucleus that is undergoing decay. Here we present a valiant attempt that illustrates a lot of the key physics, but necessarily involves skipping steps and making simplifications that are not made in modern theoretical calculations. At the nucleon level, the Hamiltonian is[1]:

$$H = \left(\sqrt{2}G_F\,|V_{ud}|\right)^2 \left[\bar{e}_L\gamma_\mu(1-\gamma_5)\nu_L\right]\left[-\bar{\nu}_L^c\gamma_\mu(1-\gamma_5)e_L^c\right]\left[\bar{p}\gamma^\mu(1-g_A\gamma_5)n\right]\left[\bar{p}\gamma^\mu(1-g_A\gamma_5)n\right] \tag{1}$$

Many elements of this object are familiar; we have the Fermi constant twice - expected for a second order weak process; the CKM matrix element $|V_{ud}|$, since the W bosons here couple u's to d's; Two terms for leptonic vertices, though one of them is a bit weird looking compared to what we might expect in a more garden-variety process due to those conjugate operations - we will be contracting this with the other term to form a propagator soon; and then two hadronic currents that turn protons into neutrons. We also see the weak vertex terms: $\gamma^\mu(1-\gamma_5)$ as expected at the leptonic vertices, but $\gamma^\mu(1-g_A\gamma_5)$ at the hadronic ones. The value of g_A is modified by nuclear effects, and depending on which nuclear formalism is being used, it may or may not already be accounted for in the nuclear matrix elements. Going from the Hamiltonian to a matrix element involving initial, final and intermediate nuclei is non-trivial, and we'll sketch it only in outline.

1.1 From Hamiltonian to matrix element

The Hamiltonian Eq. 1 can be taken as being a reasonable one for both two-neutrino and neutrinoless decays. In the case of neutrinoless double beta decay, the neutrino that is emitted at one vertex is absorbed at the other one. In quantum field theory this corresponds to replacing the two ν fields in the above expression with a fermionic propagator, and it turns out that the propagator here is the same one we would expect for Dirac fermions (See Ref 1):

$$\langle \nu_L(x)\bar{\nu}_L(y)\rangle \to \frac{1}{(2\pi)^4}\int d^4k \frac{\not{k}+m}{k^2-m^2}e^{-ik(x-y)} \tag{2}$$

The "propagator" for the nucleon part, on the other hand, is a sum over intermediate nuclear states n, and we'll mostly kick it out into a nuclear matrix element later anyway:

$$\left[\bar{p}\gamma^\mu(1-g_A\gamma_5)n\right]\left[\bar{p}\gamma^\mu(1-g_A\gamma_5)n\right] \to \sum_n \langle N_i|J^\mu|n\rangle\langle n|J^\nu|N_f\rangle \tag{3}$$

The relevant matrix element for the process ultimately takes the form:

$$\mathcal{M} = -2G_F^2\,|V_{ud}|^2\sum_n\int \frac{d^4k}{(2\pi)^4}\bar{u}(p_1)\frac{1-\gamma_5}{2}\gamma_\mu\frac{\not{k}_\nu+m}{k_\nu^2-m^2}\gamma_\nu\frac{1-\gamma_5}{2}v(p_2)e^{i\vec{k}_\nu\cdot\vec{r}}\times \tag{4}$$

$$\langle N_i|J^\mu|n\rangle\langle n|J^\nu|N_f\rangle 2\pi\delta(k_\nu - E_{1i}+E_{1n}+\epsilon_1) \tag{5}$$

the labels here are:

- k_ν : neutrino momentum

[1]Following Fukugita, Masataka, and Tsutomu Yanagida. Physics of Neutrinos: and Application to Astrophysics. Springer Science & Business Media, 2013, and Haxton and Stephenson, Nuclear Physics, Vol. 12*, 409-479

- E_{1i} : nucleon 1 initial energy
- E_{1n}: nucleon 1 intermediate state energy
- ϵ_1: first electron energy

It is challenging to evaluate the sum over intermediate states that features in the matrix element, because the matrix element is an expression of the form

$$\mathcal{M} \sim \sum_n \langle N_f|J_1|n\rangle\langle n|J_2|N_i\rangle A(E_{1n}) \tag{6}$$

Where $A(E_n)$ is a function of the intermediate state energy, and $\langle N_f|J_1|n\rangle$ are complex objects involving overlap of various nuclear wave functions, which are between difficult and impossible to calculate, for the heavy nuclei involved in $0\nu\beta\beta$. However, if we make the approximation that $A(E_{1n}) \sim A(\langle E_{int}\rangle)$, the sum becomes more manageable, only acting on the current matrix elements:

$$\mathcal{M} \to \left(\sum_n \langle N_f|J_1|n\rangle\langle n|J_2|N_i\rangle\right) A(\langle E_{int}\rangle) \to \langle N_f|J_1 \left(\sum_n |n\rangle\langle n|\right) J_2|N_i\rangle A(\langle E_{int}\rangle) \tag{7}$$

And we also note that, since the sum is over a complete set of states, $\sum_n |n\rangle\langle n| = 1$, we get

$$= \langle N_f|J_1 J_2|N_i\rangle A(\langle E_{int}\rangle) \tag{8}$$

It is this approximation that allows us to fully factorize the matrix element into independent hadronic and leptonic tensors:

$$\mathcal{M} = -2G_F^2 \left|V_{ud}\right|^2 \sum_n L^n_{\mu\nu} H^{\mu\nu}_n \to -2G_F^2 \left|V_{ud}\right|^2 L_{\mu\nu} H^{\mu\nu} \tag{9}$$

And so the spin-summed matrix element squared needed to calculate a decay rate has the form:

$$\langle|\mathcal{M}|^2\rangle = 4G_F^4 \left|V_{ud}\right|^4 \left(\sum_{spins} L_{\sigma\rho} L_{\mu\nu}\right) \left(H^{\sigma\rho} H^{\mu\nu}\right) \tag{10}$$

The leptonic part has a form that looks like it should be manageable:

$$L^{\mu\nu} = \int \frac{d^4k_\nu}{(2\pi)^4} \bar{u}(p_1) \frac{1-\gamma_5}{2} \gamma_\mu \frac{\not{k}_\nu + m_\nu}{k_\nu^2 - m_\nu^2} \gamma_\nu \frac{1-\gamma_5}{2} v(p_2) e^{i\vec{k}_\nu . r} \tag{11}$$

Considering for a moment the two added terms in the propagator, we see that the two chiral projectors force us to keep only the m_ν term, since:

$$\frac{1-\gamma_5}{2}\gamma_\mu \not{k}_\nu \gamma_\nu \frac{1-\gamma_5}{2} = \gamma_\mu \frac{1+\gamma_5}{2} \not{k}_\nu \gamma_\nu \frac{1-\gamma_5}{2} \tag{12}$$

$$= \gamma_\mu \not{k}_\nu \frac{1-\gamma_5}{2} \gamma_\nu \frac{1-\gamma_5}{2} \tag{13}$$

$$= \gamma_\mu \not{k}_\nu \gamma_\nu \frac{1+\gamma_5}{2} \frac{1-\gamma_5}{2} = 0 \tag{14}$$

So, we find:

$$L^{\mu\nu} = m_\nu \int \frac{d^4k}{(2\pi)^4} \frac{1}{k_\nu^2 - m_\nu^2} \bar{u}(p_1) \gamma_\mu \gamma_\nu \frac{1-\gamma_5}{2} v(p_2) e^{i\vec{k}_\nu . r} \tag{15}$$

Note that if $m_\nu = 0$ then $L^{\mu\nu} = 0$ and the process cannot go, as expected. Next we can use momentum balance in the denominator of the propagator to set:

$$k_\nu^2 - m_\nu^2 = (p_{1i} - p_{int} - p_1)^2 - m_\nu^2 \tag{16}$$

Continuing to re-organize:

$$= (E_{1i} - E_{int} - E_1)^2 - (\vec{p}_{1i} - \vec{p}_{int} - \vec{p}_1)^2 - m_\nu^2 \tag{17}$$

By momentum balance, $\vec{p}_{1i} - \vec{p}_{int} - \vec{p}_1$ is the momentum of the neutrino, $\vec{k}_\nu$, so

$$= (E_{1i} - E_{int} - E_1)^2 - \epsilon_\nu^2 \tag{18}$$

Where here ϵ_ν is the energy of the virtual neutrino. In continuing to manipulate the leptonic current, we can use the identity

$$\gamma^\mu \gamma^\nu = \eta^{\mu\nu} + \frac{1}{2}\sigma^{\mu\nu} \qquad (\sigma^{\mu\nu} = \frac{1}{2}[\gamma^\mu, \gamma^\nu]) \tag{19}$$

$$L_n^{\mu\nu} = m_\nu \int \frac{d^4k}{(2\pi)^4} \frac{1}{k_\nu^2 - m_\nu^2} \bar{u}(p_1) \left(\eta^{\mu\nu} + \sigma^{\mu\nu}\right) \frac{1-\gamma_5}{2} v(p_2) e^{i\vec{k}_\nu . r} \tag{20}$$

For super-allowed transitions, $0^+ \to 0^+$ there is no "magnetic" contribution and so we can drop the term proportional to $\sigma^{\mu\nu}$. This leaves us with:

$$L_n^{\mu\nu} = m_\nu \eta^{\mu\nu} \left\{ \bar{u}(p_1) \frac{1-\gamma_5}{2} v(p_2) \right\} \left[\int \frac{d^4k}{(2\pi)^4} \frac{e^{i\vec{k}_\nu . r}}{(E_{1i} - E_{int} - E_1)^2 - \epsilon_\nu^2} \right] \tag{21}$$

Using the spinor completeness relations we can now take care of the spinor part of the sum,

$$\sum_{spins} \left\{ \bar{u}(p_1) \frac{1-\gamma_5}{2} v(p_2) \right\} \left\{ \bar{u}(p_1) \frac{1-\gamma_5}{2} v(p_2) \right\}^\dagger = \mathrm{Tr} \left[(\not{p}_1 + m_e) \left(\frac{1+\gamma^5}{2} \right) (\not{p}_2 - m_e) \left(\frac{1-\gamma^5}{2} \right) \right] \tag{22}$$

In this trace, only the elements with both p's survive:

$$= 4 g_{\alpha\beta} p_{1\alpha} p_{2\beta} \tag{23}$$

This can then be used to evaluate the matrix element:

$$\langle |\mathcal{M}|^2 \rangle = 4 m_\nu^2 G_F^2 \left|V_{ud}\right|^4 H_\mu^\mu H_\rho^\rho \left[\frac{1}{4\pi} F(r) \right]^2 2 p_1 . p_2 \tag{24}$$

This is a schematic expression for the matrix element if we imagine there is only one kind of neutrino participating, with mass m_ν.

2 Multiple massive neutrinos in the three flavor paradigm

For each neutrino, there is an amplitude contribution that will involve two factors of the PMNS matrix, and the neutrino mass (collecting into X everything that is not $m_{\nu i}$ of line 24):

$$M_i = X \sum_i (U_{ei})^2 m_{\nu i} \tag{25}$$

And so the total matrix element will be the sum of these contributions, which ultimately gets squared in Fermi's Golden Rule for the decay rate:

$$\langle |\mathcal{M}|^2 \rangle = \left| X \sum_i (U_{ei})^2 m_{\nu i} \right|^2 = |X|^2 m_{\beta\beta}^2 \tag{26}$$

Above we have introduced the important effective parameter $m_{\beta\beta}$:

$$m_{\beta\beta} = \sum_i (U_{ei})^2 m_{\nu i} \tag{27}$$

And the spin-summed matrix element takes the form

$$\langle |\mathcal{M}|^2 \rangle = 4 m_{\beta\beta}^2 G_F^2 \left|V_{ud}\right|^4 H_\mu^\mu H_\rho^\rho \left[\frac{1}{4\pi} F(r) \right]^2 2 p_1 . p_2 \tag{28}$$

3 The decay rate and phase space factor

To turn the matrix element into a decay rate you need to use Fermi golden rule with all its phase space factors and delta functions, and all that. Noting that the product of momenta above is a four-vector product,

$$\frac{d\Gamma}{d\cos\theta\, dE_1} = G_F^4 \left|V_{ud}\right|^4 \left\{ \frac{1}{16\pi^5} |\vec{p}_1||\vec{p}_2| p_1.p_2 \right\} \left\{ H^\mu_\mu H^\rho_\rho \left[\frac{1}{4\pi} F(r) \right]^2 \right\} \left\{ m^2_{\beta\beta} \right\} \tag{29}$$

And so

$$\Gamma = G_F^4 \left|V_{ud}\right|^4 \left\{ \int d\cos\theta dE_1 \frac{1}{16\pi^5} |\vec{p}_1||\vec{p}_2| \left(1 - \frac{\vec{p_1}.\vec{p_2}}{E_1 E_2} \right) \right\} \left\{ H^\mu_\mu H^\rho_\rho \left[\frac{1}{4\pi} F(r) \right]^2 \right\} \left\{ m^2_{\beta\beta} \right\} \tag{30}$$

The curly bracketed pieces are called respectively the "Phase Space Factor" (G), the "Nuclear Matrix Element" ($\|M\|$), and the "Effective Majorana Mass" ($m^2_{\beta\beta}$). This is often written in compact form:

$$\Gamma = G \left\|M\right\|^2 m^2_{\beta\beta} \tag{31}$$

Occaisionally people will absorb the couplings and CKM elements into G, as:

$$G = G_F^4 \left|V_{ud}\right|^4 \tilde{G} \tag{32}$$

In the simplest version of the calculation of $\tilde{G}$, the kinematic part of G, we find:

$$\tilde{G} = \int \frac{1}{16\pi^5} dE_1 d\cos\theta\, E_1 E_2 |\vec{p}_1||\vec{p}_2| \left(1 - \frac{\vec{p_1}.\vec{p_2}}{E_1 E_2} \right) \tag{33}$$

This is an integral we can evaluate. First, we explicitly include the opening angle:

$$= \int \frac{1}{16\pi^5} dE_1 d\cos\theta\, E_1 E_2 |\vec{p}_1||\vec{p}_2| \left(1 - \frac{p_1 p_2}{E_1 E_2} \cos\theta \right) \tag{34}$$

And note that the integral $\int_{-1}^{1} d\cos\theta\, \cos\theta = 0$, so only the left term survives the angular integration:

$$= \frac{1}{8\pi^5} \int_{m_e}^{m_e+T_0} dE_1 E_1 E_2 |\vec{p}_1||\vec{p}_2| \tag{35}$$

Here T_0 is the maximum allowed kinetic energy of electron 1, and hence $E_2 = T_0 + 2m_e - E_1$. Writing everything in terms of electron energies,

$$= \frac{1}{8\pi^5} \int_{m_e}^{m_e+T_0} dE_1 E_1 \left(T_0 + 2m_e - E_1 \right) \sqrt{E_1 - m_e^2} \sqrt{\left(T_0 + 2m_e - E_1 \right)^2 - m_e^2} \tag{36}$$

This integral can be done either with blood, sweat and tears, or with Mathematica, with the result that:

$$= \frac{m_e^5}{8\pi^5} \left(\frac{t_0^5}{30} + \frac{t_0^4}{3} + \frac{4t_0^3}{3} + 2t_0^2 + t_0 \right) \qquad t_0 = \frac{T_0}{m_e} \tag{37}$$

A relatively simple answer emerges to a very complicated question. This is not quite the end of the story, however, since in the full calculation we must also include the Fermi function $F(E, Z)$ to account for the Coulomb attraction of the electron leaving the nucleus. This correction accounts for the fact that the wave function of an electron leaving as a plane wave is distorted by Coulomb attraction at the origin. The corrected expression for $\tilde{G}$ is:

$$\tilde{G} = \int \frac{1}{16\pi^5} F(E_1, Z) F(E_2, Z) dE_1 d\cos\theta\, E_1 E_2 p_1 p_2 \left(1 - \frac{\vec{p_1}.\vec{p_2}}{E_1 E_2} \right) \tag{38}$$

These coulomb effects must include relativistic corrections, and effects of shielding of the atomic electrons, which makes the integral rather more complex, though it can be evaluated using numerical methods.

4 The effective Majorana mass and the lobster plot

More interesting than the phase space factor is the effective mass that shows up in double beta decay rate of Eq. 28 is a weighted sum of the three neutrinos masses, each contributing proportionally to their probabilistic weight within the electron neutrino flavor state:

$$m_{\beta\beta} = \sum_i (U_{ei})^2 m_i \tag{39}$$

Note how this is different to the effective mass that appears in direct neutrino mass searches

$$m_\nu^2 = \sum_i |U_{ei}|^2 m_i^2 \tag{40}$$

That modulus makes all the difference, because in the direct neutrino mass case the various terms add in magnitude (so they always add up), whereas in the 0nubb case they add in amplitude (so they can cancel). We can express the matrix elements U_{ei} in terms of the mixing angles and phases that traditionally parametrize the PMNS matrix, to find:

$$m_{\beta\beta} = c_{12}^2 c_{13}^2 e^{2i\lambda_1} m_1 + c_{13}^2 s_{12}^2 e^{2i\lambda_b} m_2 + s_{13}^2 m_3 \tag{41}$$

Regarding the neutrino masses, all we know today are the their squared differences, accessed through oscillations. We do not know the absolute mass scale (the lightest m_i) or whether the observed bigger splitting is between the heaviest two or lightest two neutrinos (the "mass ordering" or "mass hierarchy"). We do know all of the mixing angles, with reasonable precision, from studies of neutrino oscillations between various flavors and on various baselines. We might now know something about δ_{CP} - if we do then this is recent news, with some controversies still unresolved. We know nothing about the Majorana phases λ_a and λ_b since they do not feature in oscillations, and we have little hope of learning about them, short of observing neutrinoless double beta decay.

In terms of these known and unknown parameters $m_{\beta\beta}$ can be expressed as:

$$m_{\beta\beta} = c_{12}^2 c_{13}^2 m_1 e^{2i\lambda_a} + s_{12}^2 c_{13}^2 e^{2i\lambda_2} \sqrt{m_1^2 + \Delta m_{12}^2} + s_{13}^2 \sqrt{m_1^2 \pm |\Delta m_{23}^2|} \tag{42}$$

The $\pm$ under the square root of Eq. 42 reflects that at the present time we know the absolute scale of Δm_{23}^2 (from atmospheric and accelerator neutrino experiments) but we do not know its sign (the "mass ordering", or "mass heirachy"). On the other hand, Δm_{12}^2 is known from solar neutrino oscillation experiments where the MSW effect would drive oscillations differently depending on the relevant ordering, so we do know both its value and sign.

Given the freedom to choose all the unknown parameters $\lambda_a, \lambda_b, m_1$ in Eq.42, as well as make one discrete choice of the sign of Δm_{23}^2, we find two wide swathes of allowed decay rates. These bands are commonly represented on what has become colloquially known as "lobster plot" of Fig. **??**, right. Here the allowed values for the parameter $m_{\beta\beta}$ featuring in the decay rate (or equivalently the lifetime) is shown with its allowed values plotted against the lightest neutrino mass. The lifetime of neutrinoless double beta decay is proportional to $|m_{\beta\beta}|^2$.