Math Interlude! Beta decay kinematics and neutrino mass

Ben Jones, National Nuclear Physics Summer School Lecture 1

Fermi's golden rule gives the rate of a beta decay as

$$\Gamma = \int \frac{d^3p_e}{(2\pi)^3}\frac{d^3p_\nu}{(2\pi)^3}\frac{d^3p_N}{(2\pi)^3}|T_{fi}|^2\delta(E_0 - E_e - E_\nu - E_N)\delta^3(\vec{p}_0 - \vec{p}_e - \vec{p}_\nu - \vec{p}_N). \tag{1}$$

Here, p_e, p_ν, p_N and E_e, E_ν, E_N are the momenta and energies of the electron, neutrino and recoiling nucleus respectively; p_0 and E_0 are the momentum and energy of the initial nucleus; $|T_{fi}|^2$ is the decay matrix element connecting the initial and final state. We can consider the initial nucleus and rest and neglect the the recoil energy of the daughter nucleus, to find

$$\Gamma = \int \frac{d^3p_e}{(2\pi)^3}\frac{d^3p_\nu}{(2\pi)^3}\frac{d^3p_N}{(2\pi)^3}|T_{fi}|^2\delta(\Delta M - E_e - E_\nu)\delta^3(\vec{p}_e + \vec{p}_\nu + \vec{p}_N). \tag{2}$$

Where ΔM is the mass difference between the parent and daughter nucleus. Since $\vec{p}_N$ now doesn't feature anywhere except in the delta function, we can integrate over it trivially, to obtain

$$\Gamma = \int \frac{d^3p_e}{(2\pi)^3}\frac{d^3p_\nu}{(2\pi)^3}|T_{fi}|^2\delta(\Delta M - E_e - E_\nu). \tag{3}$$

The matrix element in beta decay has the general form:

$$|T_{fi}|^2 = F(Z,E)G_F^2|V_{ud}|^2\,|\mathcal{T}_{fi}|^2. \tag{4}$$

The factors in this expression are the Fermi constant G_F, an element of the CKM matrix $|V_{ud}|^2$, the matrix element calculated from spin-structure the nuclear wave function $\mathcal{T}_{fi}$ which is a dimensionless number that varies slowly with energy, so can be approximated as (roughly) constant for the decay, and something called the "Fermi Function", $F(Z,E)$ that accounts for the electromagnetic interaction of the outgoing electron with the nucleus. We'll ignore the Fermi function today.

1 Kurie plot with zero neutrino mass

When measuring the beta spectrum, we measure the emitted electron, and can thus can experimentally access the differential decay rate $\frac{d\Gamma}{dE_e}$. This can be extracted from Γ as expressed below by turning the electron momentum integral into an electron energy integral, in the following steps:

$$\Gamma = \int \frac{d^3p_e}{(2\pi)^3}\frac{d^3p_\nu}{(2\pi)^3}|T_{fi}|^2\delta(\Delta M - E_e - E_\nu), \tag{5}$$

$$= \int d^3p_e \int p_\nu^2 d\Omega_\nu dp_\nu \frac{1}{(2\pi)^6}|T_{fi}|^2\delta(\Delta M - E_e - E_\nu). \tag{6}$$

If the neutrino is in fact massless then $E_\nu = p_\nu$, and

$$\Gamma = \int d^3p_e \int p_\nu^2 d\Omega_\nu dp_\nu \frac{1}{(2\pi)^6}|T_{fi}|^2\delta(\Delta M - E_e - p_\nu), \tag{7}$$

$$= \int dp_e \left[\frac{1}{2\pi^3}p_e^2(\Delta M - E_e)^2|T_{fi}|^2\right]. \tag{8}$$

What we have here is an integral over a differential decay rate in p_e,

$$\Gamma = \int dp_e \frac{d\Gamma}{dp_e} \qquad \frac{d\Gamma}{dp_e} = \frac{1}{2\pi^3}p_e^2(\Delta M - E_e)^2|T_{fi}|^2. \tag{9}$$

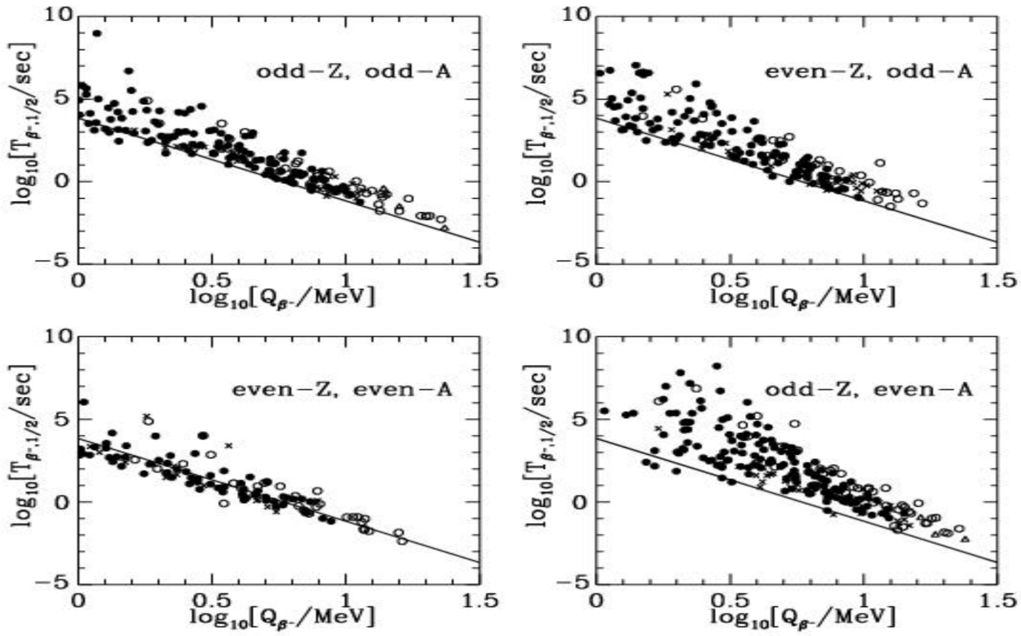

Figure 1: Approximate Q^5 dependence of beta decay

From this we can acquire $\frac{d\Gamma}{dE_e}$ by change of variables,

$$\frac{d\Gamma}{dE_e} = \frac{dp_e}{dE_e}\frac{d\Gamma}{dp_e}. \tag{10}$$

Since $E^2 = p^2 + m^2$, we have $EdE = pdP$ and $\frac{dp_e}{dE_e} = \frac{E_e}{p_e}$, leading to

$$\frac{d\Gamma}{dp_e} = \frac{1}{2\pi^3}p_e E_e(\Delta M - E_e)^2|T_{fi}|^2. \tag{11}$$

Substituting in for $|T_{fi}|^2$, we find

$$\frac{d\Gamma}{dE_e} = F(Z,E)\frac{G_F^2|V_{ud}|^2}{2\pi^3}|\mathcal{T}_{fi}|^2 p_e E_e(\Delta M - E_e)^2. \tag{12}$$

Note that all of E_e, p_e, E_0 and the range of integration over which $\frac{d\Gamma}{dE}$ must be integrated are of order $Q = \Delta M - m_e$, and so one expects the rate of β decay to be roughly proportional to Q^5. This effect, simply a consequence of phase space arguments, is indeed observed for a large class of decays, as shown in the figure above.

A standard way to analyze the shape of the spectrum is via a Kurie plot. This plot involves plotting the function $K(E_e)$ against electron energy, where:

$$K(E_e) = \left[\frac{\frac{d\Gamma}{dE_e}}{F(Z,E)E_e p_e}\right]^{1/2}. \tag{13}$$

If the nuclear matrix element $\mathcal{T}_{fi}$ had a non-trivial dependence on energy, this function could assume any shape. However, if $\mathcal{T}_{fi}$ were independent of energy, this function would have the form

$$K(E_e) \propto (\Delta M - E_e). \tag{14}$$

This is a straight line intersecting the origin at the Q-value of the decay. For many nuclei this is exactly what is observed. For example, Fig. 2, left shows a Kurie plot for ^{187}Re - a beautiful, straight line.

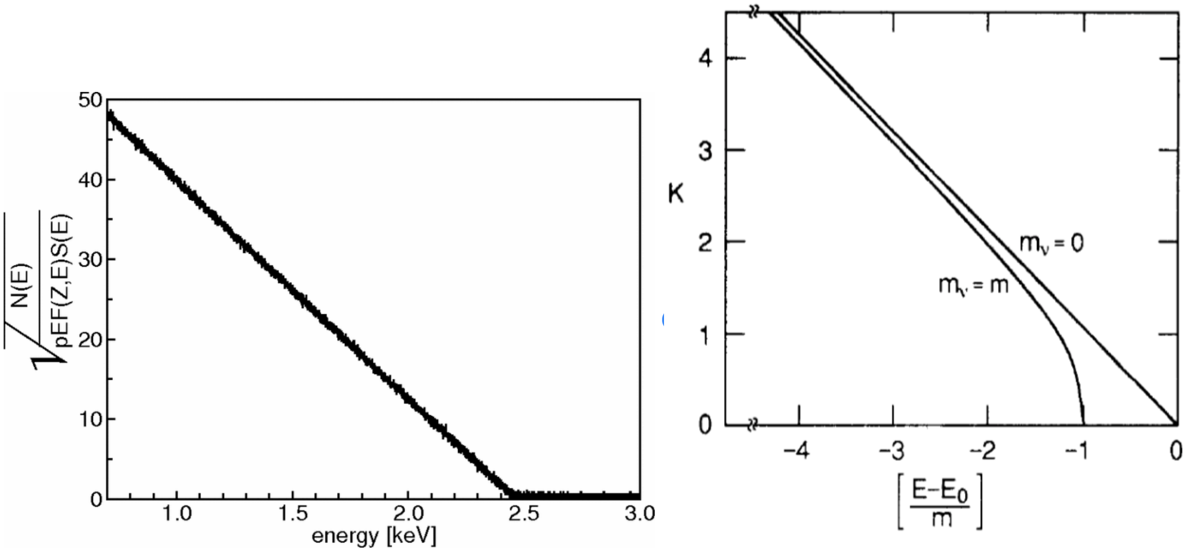

Figure 2: Left: Kurie plot from the MARE collaboration studying β decay of ^{187}Re. Right: method of measuring neutrino mass from distortion in the end-point of the Kurie plot of tritium (used for its very low Q-value)

2 Kurie plot with nonzero neutrino mass

Now, what if the neutrino mass were non-zero? This changes our math a little bit, starting at Eq. 6

$$\Gamma = \int d^3p_e \int p_\nu^2 d\Omega_\nu dp_\nu \frac{1}{(2\pi)^6} |T_{fi}|^2 \delta(\Delta M - E_e - E_\nu), \tag{15}$$

$$= \int d^3p_e \int p_\nu^2 d\Omega_\nu dp_\nu \frac{1}{(2\pi)^6} |T_{fi}|^2 \delta(\Delta M - E_e - \sqrt{p_\nu^2 + m_\nu^2}) \tag{16}$$

Using the the delta function integration trick, $\int dx\, \delta(f(x)) = \int dx \frac{\delta(x-x_0)}{|df/dx|}$,

$$= \int d^3p_e \int d\Omega_\nu \frac{1}{(2\pi)^6} p_\nu^2 \frac{\sqrt{p_{\nu 0}^2 + m_\nu^2}}{p_{\nu 0}} |T_{fi}|^2 = \int d^3p_e \int d\Omega_\nu \frac{1}{(2\pi)^6} p_{\nu 0} E_{\nu 0} |T_{fi}|^2 \tag{17}$$

In the above expression, p_ν and E_ν are now fixed at its value imposed by the delta function integration,

$$p_{\nu 0} = \sqrt{(\Delta M - E)^2 - m_\nu^2} \tag{18}$$

. Thus:

$$\Gamma = \frac{1}{2\pi^3} \int dp_e p_e^2 \sqrt{(\Delta M - E_e)^2 - m_\nu^2} (\Delta M - E_e) |T_{fi}|^2 \tag{19}$$

Everything now proceeds as it did before, just carrying along that extra square root term; we ultimately arrive at a Kurie function that goes like

$$K(E) \propto \left[(\Delta M - E_e) \sqrt{(\Delta M - E_e)^2 - m_\nu^2} \right]^{1/2}. \tag{20}$$

How has this changed things? Well, when $\Delta M - E_e \gg m_\nu^2$, which is true almost everywhere in the spectrum, the answer is, it doesn't really change them at all; the expression reduces to the one we had for $m_\nu = 0$. But, very near the end-point when the electron energy is close to the end-point energy, the spectrum turns downward. One way to think of this suppression is that, since the decay has to produce a neutrino with finite mass, there not enough energy available to push the electron all the way up to the end-point.

3 Multiple massive neutrinos

In real life, we don't have just one kind of neutrino, we have three; the electron neutrino emitted in beta decay is actually a quantum superposition of each of the three neutrino mass eigenstates,

$$|\nu_e\rangle = \sum_i U_{ei}|\nu_i\rangle. \tag{21}$$

Each ν_i has mass m_i. How should we account for this in the decay rate calculation? First, we recognize that because the final state is distinguishable for each kind of neutrino $N_0 \rightarrow N + e + \bar{\nu}_i$, we should add the rates for each mass state incoherently (that is, add the rates, not add the amplitudes), and so

$$\frac{d\Gamma}{dE_e} = \sum_i \frac{d\Gamma}{dE_{e,i}}. \tag{22}$$

Here, $\frac{d\Gamma}{dE_{e,\nu_i}}$ is the rate of decay to an electron of energy E_i accompanied by neutrino mass state $\bar{\nu}_i$. Performing the sum over mass states,

$$\frac{d\Gamma}{dE_e} = F(Z,E)\frac{G_F^2|V_{ud}|^2}{2\pi^3}|\mathcal{T}_{fi}|^2 p_e E_e(\Delta M - E_e)\left(\sum_i |U_{ei}|^2\sqrt{(\Delta M - E_e)^2 - m_i^2}\right). \tag{23}$$

There is a little manipulation that can be applied to the term in brackets, which works only in the part of the spectrum where $\Delta M - {E_e}^2 \gg m_i^2$, for all of the m_i^2. In that "degenerate region", we can Taylor expand the square root, and

$$\left(\sum_i |U_{ei}|^2\sqrt{(\Delta M - E_e)^2 - m_i^2}\right) \sim \sum_i |U_{ei}|^2\left(\sqrt{(\Delta M - E_e)^2} - \frac{m_i^2}{\sqrt{(\Delta M - E_e)^2}}\right) \tag{24}$$

$$= \left(\sqrt{(\Delta M - E_e)^2}\sum_i |U_{ei}|^2 - \frac{\sum_i |U_{ei}|^2 m_i^2}{\sqrt{(\Delta M - E_e)^2}}\right) \tag{25}$$

Since by unitarity of the PMNS matrix $\sum_i |U_{ei}|^2 = 1$ it drops out of the first term, and so,

$$= \left(\sqrt{(\Delta M - E_e)^2} - \frac{\sum_i |U_{ei}|^2 m_i^2}{\sqrt{(\Delta M - E_e)^2}}\right) \tag{26}$$

And we can then "Taylor un-expand" again, to find approximate equivalence to

$$\sim \left(\sqrt{(\Delta M - E_e)^2 - m_\nu^2}\right). \tag{27}$$

m_ν is the effective electron neutrino mass, defined as

$$m_\nu = \sum_i |U_{ei}|^2 m_i^2 \tag{28}$$

This means that a massive and mixed neutrino distorts the end point of the beta spectrum in a similar way to a single neutrino with the "PMNS-averaged" mass of its components.

$$\frac{d\Gamma}{dE_e} = F(Z,E)\frac{G_F^2|V_{ud}|^2}{2\pi^3}|\mathcal{T}_{fi}|^2 p_e E_e(\Delta M - E_e)\left(\sqrt{(\Delta M - E_e)^2 - m_\nu^2}\right) \tag{29}$$

There is one caveat to give, which is that if we are not considering a part of the spectrum where the approximation $\Delta M - {E_e}^2 \gg m_i^2$ is valid, this formulation is not appropriate. In that regime, if we look closely enough at the spectrum, features associated with each mass of neutrino will generating a distinct kink. Those kinks are very challenging to see for the normal active neutrinos; but in the case of heavy sterile neutrinos, for example, they may appear further from the end point in an observable way. Experiments such as KATRIN perform analyses looking for these kinks, setting useful limits on additional heavy neutrinos that could mix with the electron neutrino, in addition to their flagship physics measurements targeting the active neutrino mass end-point effect embodied in m_ν. In such a scenario, the full expression Eq. 23 should be used, rather than analyzing in terms of the effective m_ν.

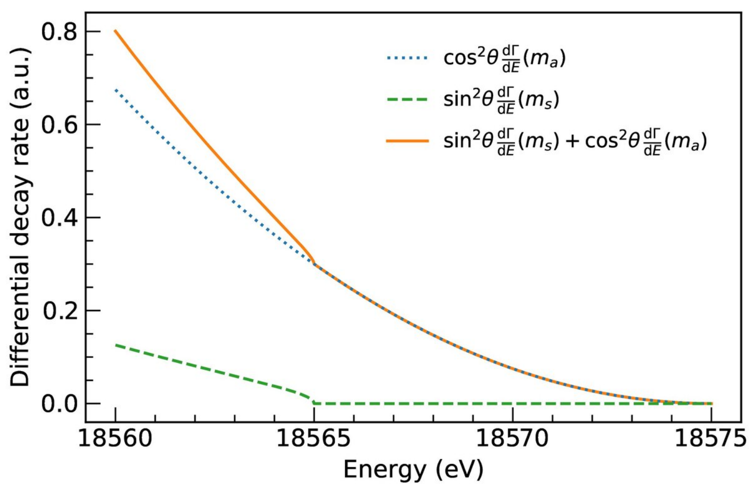

Figure 3: Electron energy spectrum in the case where one neutrino mass is significantly heavier (e.g. Katrin sterile neutrino search)