Neutrino Oscillations and CP Violation

Ben Jones, National Nuclear Physics Summer School Lecture 4

1 Oscillations in three flavors

Three flavor oscillations is an almost trivial extension of the two-flavor proof, but it requires carrying more parameters around. Now we have a three dimensional PMNS matrix,

$$\begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix} = \begin{pmatrix} U_{e1} & U_{e2} & U_{e3} \\ U_{\mu 1} & U_{\mu 2} & U_{\mu 3} \\ U_{\tau 1} & U_{\tau 2} & U_{\tau 3} \end{pmatrix} \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix} \tag{1}$$

The most general unitary 3x3 matrix is more complicated than the most general unitary 2x2 matrix. In particular, it requires multiple mixing angles is allowed to have up to three CP-violating phases (of which two can be parameterized out by rephasing neutrino fields, if the neutrino is a Dirac particle). The standard parameterization is

$$U^{3x3}_{PMNS} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix} \begin{pmatrix} c_{13} & 0 & s_{13}e^{-i\delta} \\ 0 & 1 & 0 \\ -s_{13}e^{-i\delta} & 0 & c_{13} \end{pmatrix} \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{i\lambda_1} & 0 \\ 0 & 0 & e^{i\lambda_2} \end{pmatrix}, \tag{2}$$

and the Hamiltonian is a 3x3 diagonal matrix in the mass basis,

$$H = \begin{pmatrix} E_1 & 0 & 0 \\ 0 & E_2 & 0 \\ 0 & 0 & E_3 \end{pmatrix}. \tag{3}$$

Rather than carrying all these parameters along, we follow matrix notation, with the added benefit that this will also work for N flavors. We start with initial state

$$|\psi(t=0)\rangle = |\nu_\alpha\rangle = U_{\alpha i}|\nu_i\rangle. \tag{4}$$

For a flavor transformation $\alpha \to \beta$, the probability $P_{\alpha\to\beta}$ is

$$P_{\alpha\to\beta} = |\langle \nu_\beta|\psi(t)\rangle|^2. \tag{5}$$

Substituting in the flavor states in terms of mass basis states,

$$= \left| \sum_{ij} \left(|\langle \nu_j| U^*_{\beta j} \right) e^{-iHt} \left(U_{\alpha i} |\nu_i\rangle \right) \right|^2, \tag{6}$$

and a little good ol' fashioned quantum mechanics elbow grease will get us to

$$= \left| \sum_i e^{-iE_i t} \left(U^*_{\beta i} U_{\alpha i} \right) \right|^2. \tag{7}$$

Since every term in this expression will have an $e^{iE_i t}$ multiplied by an $e^{-iE_j t}$ for some i, j, we can subtract an arbitrary constant from the energies (and hence from the Hamiltonian), without changing the oscillations. It is thus convenient to subtract the energy that would be associated with a massless neutrino, as an overall constant. In this case, the Hamiltonian matrix becomes

$$H \to \begin{pmatrix} E_1 & 0 & 0 \\ 0 & E_2 & 0 \\ 0 & 0 & E_3 \end{pmatrix} - \begin{pmatrix} E_0 & 0 & 0 \\ 0 & E_0 & 0 \\ 0 & 0 & E_0 \end{pmatrix} = \begin{pmatrix} \frac{m_1^2}{2E} & 0 & 0 \\ 0 & \frac{m_2^2}{2E} & 0 \\ 0 & 0 & \frac{m_3^2}{2E} \end{pmatrix}. \tag{8}$$

The expression above thus becomes

$$P_{\alpha\to\beta} = \left| \sum_i e^{-i\frac{m_i^2 L}{2E}} \left(U_{\beta i}^* U_{\alpha i}\right) \right|^2. \tag{9}$$

Which is equivalent to

$$= \sum_{i,j} \left(U_{\beta i} U_{\alpha i}^* U_{\beta j}^* U_{\alpha j}\right) e^{-i\frac{\Delta m_{ij}^2 L}{2E}} \tag{10}$$

2 C, P, CP and CPT in neutrino oscillations

Consider a neutrino oscillation probability between two flavors, α and β. We note that neutrinos are always left-handed, so its a bit superfluous to write it down, but we'll put an "L" index to remind us,

$$P_{\nu_\alpha L\to\nu_\beta L} = \sum_{i,j} \left(U_{\beta i} U_{\alpha i}^* U_{\beta j}^* U_{\alpha j}\right) e^{-i\frac{\Delta m_{ij}^2 L}{2E}} \tag{11}$$

We are now going to consider what the effects of the discrete transformations C, P, CP and CPT on this probability. C is the operation that transforms particles into antiparticles; P is parity, which transforms left-spinning particles to right-spinning ones; and T is time reversal, which reverses the order of the oscillation. The first thing we note is that each of C and P acting alone leads us to an unphysical place, since as far as we know, right handed neutrinos and left handed antineutrinos do not exist:

$$P_{\nu_\alpha L\to\nu_\beta L} \quad \underset{P}{\to} \quad P_{\nu_\alpha R\to\nu_\beta R} \tag{12}$$

$$P_{\nu_\alpha L\to\nu_\beta L} \quad \underset{C}{\to} \quad P_{\bar\nu_\alpha L\to\bar\nu_\beta L} \tag{13}$$

So, neutrino oscillations surely violate C and P symmetry since those transformations lead to unphysical processes. This is fine, neutrinos are made in weak interactions and weak interactions are maximally parity violating and near-maximally C violating.

C and P performed together, do transform an in-principle physical oscillation probability into another in-principle physical one,

$$P_{\nu_\alpha L\to\nu_\beta L} \quad \underset{CP}{\longrightarrow} \quad P_{\bar\nu_\alpha R\to\bar\nu_\beta R}. \tag{14}$$

The question of whether neutrino oscillations are CP-violating is therefore the question of whether these two are the same:

$$P_{\nu_\alpha\to\nu_\beta} \quad =^? \quad P_{\bar\nu_\alpha\to\bar\nu_\beta} \tag{15}$$

IE, do neutrinos (which are necessarily left handed) and antineutrinos (which are necessarily right handed) oscillate the same or differently?

All Lorentz-invariant quantum field theories are CPT invariant, thus it would be highly surprising if neutrinos were to violate CPT. Under CPT the oscillation probability becomes

$$P_{\nu_\alpha\to\nu_\beta} \quad \underset{CPT}{\longrightarrow} \quad P_{\bar\nu_\beta\to\bar\nu_\alpha} \tag{16}$$

This enables for a direct test of CPT violation in oscillations by measuring the probabilities in appropriate pairs of channels. As we'll see, we cannot encode CPT effects in the mixing matrix, they would break the theory at a deeper level.

3 CP violation and the Jarlskog invariant

We are interested to explore the question, do neutrinos and antineutrinos oscillate differently? If they do, this is evidence of leptonic CP violation. To dig into this question it helps to break the sum in $\mathcal{P}$ into two parts,

$$P_{\alpha\to\beta} = \sum_{i=j} \left(U_{\beta i}U^*_{\alpha i}U^*_{\beta j}U_{\alpha j}\right) + \sum_{i>j} \left[\left(U_{\beta i}U^*_{\alpha i}U^*_{\beta j}U_{\alpha j}\right) e^{-i\frac{\Delta m^2_{ij}L}{2E}} + \left(U_{\beta j}U^*_{\alpha j}U^*_{\beta i}U_{\alpha i}\right) e^{i\frac{\Delta m^2_{ij}L}{2E}}\right] \tag{17}$$

The left term is equal to $\delta_{\alpha\beta}$, through the unitarity of the CKM matrix. The right term can be written as a real part,

$$= \delta_{\alpha\beta} + 2\sum_{i>j} \mathcal{R}\left[\left(U_{\beta i}U^*_{\alpha i}U^*_{\beta j}U_{\alpha j}\right) e^{-i\frac{\Delta m^2_{ij}L}{2E}}\right] \tag{18}$$

A real number can be either a product of two reals of the product of two imaginaries,

$$= \delta_{\alpha\beta} + 2\sum_{i>j} \left[\mathcal{R}\left(U_{\beta i}U^*_{\alpha i}U^*_{\beta j}U_{\alpha j}\right) \mathcal{R}e^{-i\frac{\Delta m^2_{ij}L}{2E}} + \mathcal{I}\left(U_{\beta i}U^*_{\alpha i}U^*_{\beta j}U_{\alpha j}\right) \mathcal{I}e^{-i\frac{\Delta m^2_{ij}L}{2E}}\right] \tag{19}$$

$$= \delta_{\alpha\beta} + 2\sum_{i>j} \left[\mathcal{R}\left(U_{\beta i}U^*_{\alpha i}U^*_{\beta j}U_{\alpha j}\right) \cos\left(\frac{\Delta m^2_{ij}L}{2E}\right) + \mathcal{I}\left(U_{\beta i}U^*_{\alpha i}U^*_{\beta j}U_{\alpha j}\right) \sin\left(\frac{\Delta m^2_{ij}L}{2E}\right)\right] \tag{20}$$

Now we use a trig formula to obtain

$$\frac{1}{2} - \frac{1}{2}\cos\left(\frac{\Delta m^2_{ij}L}{2E}\right) = \sin^2\left(\frac{\Delta m^2_{ij}L}{4E}\right) \tag{21}$$

So

$$= \delta_{\alpha\beta} + 2\sum_{i>j} \left[\mathcal{R}\left(U_{\beta i}U^*_{\alpha i}U^*_{\beta j}U_{\alpha j}\right) \left(2 + 2\sin^2\left(\frac{\Delta m^2_{ij}L}{2E}\right)\right) + \mathcal{I}\left(U_{\beta i}U^*_{\alpha i}U^*_{\beta j}U_{\alpha j}\right) \sin\left(\frac{\Delta m^2_{ij}L}{2E}\right)\right] \tag{22}$$

Unitarity ensures that $\sum_i U_{\beta i}U^*_{\beta j} = 0$ for $\alpha \neq \beta$, so

$$P_{\nu_\alpha\to\nu_\beta} = \delta_{\alpha\beta} + 4\sum_{i>j} \mathcal{R}\left(U_{\beta i}U^*_{\alpha i}U^*_{\beta j}U_{\alpha j}\right) \sin^2\left(\frac{\Delta m^2_{ij}L}{2E}\right) + 2\sum_{i>j} \mathcal{I}\left(U_{\beta i}U^*_{\alpha i}U^*_{\beta j}U_{\alpha j}\right) \sin\left(\frac{\Delta m^2_{ij}L}{2E}\right) \tag{23}$$

One of the reasons this form of the expression is convenient is that it explicitly separates the CP-violating and non-CP-violating terms. If there is no oscillation-observable CP-violation, that is, if $\delta = 0$, the right term is zero. Thus this effect contains the CP violating part of the oscillation. To consider the effect of oscillations of antineutrinos rather than neutrinos, take each $U \to U^*$.

$$P_{\bar\nu_\alpha\to\bar\nu_\beta} = \delta_{\alpha\beta} + 4\sum_{i>j} \mathcal{R}\left(U^*_{\beta i}U_{\alpha i}U_{\beta j}U^*_{\alpha j}\right) \sin^2\left(\frac{\Delta m^2_{ij}L}{2E}\right) + 2\sum_{i>j} \mathcal{I}\left(U^*_{\beta i}U_{\alpha i}U_{\beta j}U^*_{\alpha j}\right) \sin\left(\frac{\Delta m^2_{ij}L}{2E}\right) \tag{24}$$

A complex conjugate of a product is equal to the product of the conjugates, so

$$P_{\bar\nu_\alpha\to\bar\nu_\beta} = \delta_{\alpha\beta} + 4\sum_{i>j} \mathcal{R}\left([U_{\beta i}U^*_{\alpha i}U^*_{\beta j}U_{\alpha j}]^*\right) \sin^2\left(\frac{\Delta m^2_{ij}L}{2E}\right) + 2\sum_{i>j} \mathcal{I}\left([U_{\beta i}U^*_{\alpha i}U^*_{\beta j}U_{\alpha j}]^*\right) \sin\left(\frac{\Delta m^2_{ij}L}{2E}\right) \tag{25}$$

And of course,

$$\mathcal{R}\left(z^*\right) = \mathcal{R}(z), \quad \mathcal{I}(z^*) = -\mathcal{I}(z), \tag{26}$$

implying that the CP-violating asymmetry is

$$P_{\nu_\alpha\to\nu_\beta} - P_{\bar\nu_\alpha\to\bar\nu_\beta} = 4\sum_{i>j} \mathcal{I}\left(U_{\beta i}U^*_{\alpha i}U^*_{\beta j}U_{\alpha j}\right) \sin\left(\frac{\Delta m^2_{ij}L}{2E}\right) \tag{27}$$

The particular combination of PMNS elements in this expression gives the "Jarlskog invariant", a generic measure of the amount of CP-violation in a mixed system,

$$\mathcal{I}\left(U_{\beta i}U_{\alpha i}^{*}U_{\beta j}^{*}U_{\alpha j}\right) = J\sum_{\gamma,l}\epsilon_{\alpha\beta\gamma}\epsilon_{jkl} \tag{28}$$

Substituting in the PMNS matrix elements we can extract the Jarlskog invariant for the three-neutrino system,

$$J = s_{13}c_{13}^{2}s_{12}c_{12}s_{23}c_{23}\sin\delta \tag{29}$$

And the amount of CP-violation in a given oscillation channel $\nu_\alpha \to \nu_\beta$ with $\beta \neq \alpha$ can be found from Eq. 27 to be

$$P_{\nu_\alpha\to\nu_\beta} - P_{\bar{\nu}_\alpha\to\bar{\nu}_\beta} = 8J\sin\left(\frac{\Delta m_{13}^2 L}{4E}\right)\sin\left(\frac{\Delta m_{23}^2 L}{4E}\right)\sin\left(\frac{\Delta m_{21}^2 L}{4E}\right) \tag{30}$$

We infer from this derivation several important conclusions about CP-violation in the neutrino oscillation system:

1. CP violation is not possible for a two-neutrino system, because the most general mixing matrix has no complex parameters.
2. CP-violation is not possible in disappearance channels where we search for $\nu_\alpha \to \nu_\alpha$, because $U_{\alpha i}U_{\alpha i}^{*}U_{\alpha j}^{*}U_{\alpha j}$ is a real number. It can only be observed in appearance channels, $\nu_\alpha \to \nu_\beta$.
3. The Majorana phases λ_1 and λ_2 introduce no observable CP-violation into neutrino oscillations.
4. If any of $\theta_{12}, \theta_{13}, \theta_{23}$ are equal to 0 (or to $\pi/2$, π or $3\pi/2$), there can be no CP-violation, due the form of Eq. 29.
5. A corollary of point (2), if any of the mixing angles are small, the observable size of CP violation in neutrino oscillations is small, even for a large value of δ_{CP}.
6. CP violation is only observable at baselines sufficiently long that multiple oscillation wavelengths are active (ie in the "three-neutrino oscillation" regime). Otherwise, one of the *sin* terms in the product 30 would be zero.

4 CPT in neutrino oscillations

Comparing the predictions for the two channels

$$P_{\nu_\alpha\to\nu_\beta} \quad \underset{CPT}{\longrightarrow} \quad P_{\bar{\nu}_\beta\to\bar{\nu}_\alpha} \tag{31}$$

We find

$$P_{\nu_\alpha\to\nu_\beta} = \delta_{\alpha\beta} + 4\sum_{i>j}\mathcal{R}\left(U_{\beta i}U_{\alpha i}^{*}U_{\beta j}^{*}U_{\alpha j}\right)\sin^2\left(\frac{\Delta m_{ij}^2 L}{2E}\right) + 2\sum_{i>j}\mathcal{I}\left(U_{\beta i}U_{\alpha i}^{*}U_{\beta j}^{*}U_{\alpha j}\right)\sin\left(\frac{\Delta m_{ij}^2 L}{2E}\right) \tag{32}$$

$$P_{\bar{\nu}_\beta\to\bar{\nu}_\alpha} = \delta_{\beta\alpha} + 4\sum_{i>j}\mathcal{R}\left(U_{\alpha i}U_{\beta i}^{*}U_{\alpha j}^{*}U_{\beta j}\right)\sin^2\left(\frac{\Delta m_{ij}^2 L}{2E}\right) - 2\sum_{i>j}\mathcal{I}\left(U_{\alpha i}U_{\beta i}^{*}U_{\alpha j}^{*}U_{\beta j}\right)\sin\left(\frac{\Delta m_{ij}^2 L}{2E}\right) \tag{33}$$

And since $U_{\alpha i}U_{\beta i}^{*}U_{\alpha j}^{*}U_{\beta j} = [U_{\beta i}U_{\alpha i}^{*}U_{\beta j}^{*}U_{\alpha j}]^{*}$ we find that no matter what the mixing parameters are,

$$P_{\nu_\alpha\to\nu_\beta} = P_{\bar{\nu}_\beta\to\bar{\nu}_\alpha} \tag{34}$$

Thus we see that CPT invariance is built into this formalism, as indeed it is built into any Lorentz invariant quantum field theory. Any observed violation of CPT would require not only modifications to the framework of neutrino oscillations but to the foundations of the standard model at large.